Cerfs Theorem

Riley Moriss

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The goal here is to understand the following statement,

$$\pi_0 \mathscr{C}_{\mathrm{id}}(M) = 0 \implies \big(\text{two diffeomorphisms are isotopic iff they are pseudo-isotopic}\big)$$

For the biconditional on the right hand side the forward implication is immediate. First lets make precise our notions of isotopy and psuedo-isotopy. [Cer71] Note that Cerf proved his theorem by first showing connectedness and then deriving the RHS.

An isotopy of diffeomorphisms is a map $f : M \times I \to M \times I$ is a diffeomorphism that commutes with the projection onto I. One can think of this as an isotopy between the diffeomorphisms of M given by restriction to the boundaries of I. A psuedo-isotopy of diffeomorphisms is a map $f : M \times I \to M \times I$ such that $f(M \times \{i\}) = M \times \{i\}$ for $i = 0, 1$. We denote by $\mathscr{C}(M)$ the space of psuedo-isotopies and $\mathscr{C}_{\mathrm{id}}$ the space of psudo-isotopies that *end at the identity*, note that this is the standard definition of $\mathscr{C}(M)$.

Because

$$[M \times I, M \times I] \cong [I, [M, M \times I]]$$

we can think of these maps as paths in the space of $\mathrm{Hom}(M, M \times I)$. We should then clarify what the conditions mean in this setting.

Lemma. *There is a bijection between paths in $[M, M]$ and paths in $[M, M \times I]$ that are the identity on the I.*

Proof. Given a path in $[M, M]$ we get a path in $[M, M \times I]$ that is the identity on the I by taking

$$\big(t \mapsto f_t : M \to M\big) \mapsto \big(t \mapsto f_t : M \to M \times I, m \mapsto (f_t(m, t)\big)$$

the inverse to this map is just composing with the projection $M \times I \to M$.

For psudo-isotopies one cannot remove the second copy of the I. The paths are merely those with the property that at $t = 0, 1$ the induced map is $f_t(m) = (f_t(m), t)$.

Lemma (Cor 1.3)**.** $\pi_0 \mathscr{C}(M) = 0$ *implies that two diffeomorphisms are isotopic if they are psuedo-isotopic.*

Proof. This is one direction of the full statement as we have pointed out that the other direction of the bi-conditional on the right is trivial.

So consider two diffeomorphisms h_1, h_0 of M and a psudeo-isotopy between them $h_t : M \times I_1 \to M \times I_1$. Becuase the space of psuedo-isotopies is connected there is a homotopy from h_t to the identity diffeomorphism $\mathrm{id}_{M \times I_1}$, or whats the same is a path in $\mathscr{C}(M)$ starting at h_t and ending at the identity, call this path $\Phi : I_2 \to \mathscr{C}(M)$. Using the adjunction we see that this path is a map

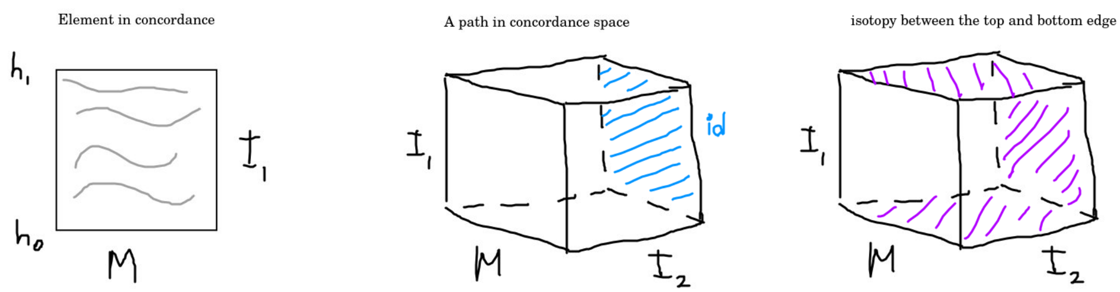
$$[I_2, [I_1, [M, M \times I_1]]] \cong [I_2 \times I_1, [M, M \times I_1]]$$

Fixing $t_1 = 0$ and then considering the resulting map

$$[I_2, [M, M]]$$

a priori results in an isotopy. There are two subtelties here. One is that we are assuming that a path in the concordance space when we apply these adjunctions results in another psuedo-isotopy, this is a matter of the topology on the concordance space. The other subtlety is that becuase we have fixed the I_1 variable it disapears from both sides of our maps and therefore gives an isotopy, now between h_0 and id. It is clear that there is nothing special about h_0 and so by symmetry we also obtain an isotopy between h_1 and the identity. Hence by the transitivity and symmetry of isotopy we have procured one from h_0 to h_1.

Note that this seems to imply that the space of isotopies is also connected, however it is actually contractable, so we are clearly on the right path. We dont claim that these images mean anything but they give a sketch of what is going on here:



It is this lemma that requires the $\mathscr{C}_{\mathrm{id}}$, without assuming that one end of the psuedo-isotopy lands at the identity we cannot concatenate them to get our path in the space of psuedo-isotopies.

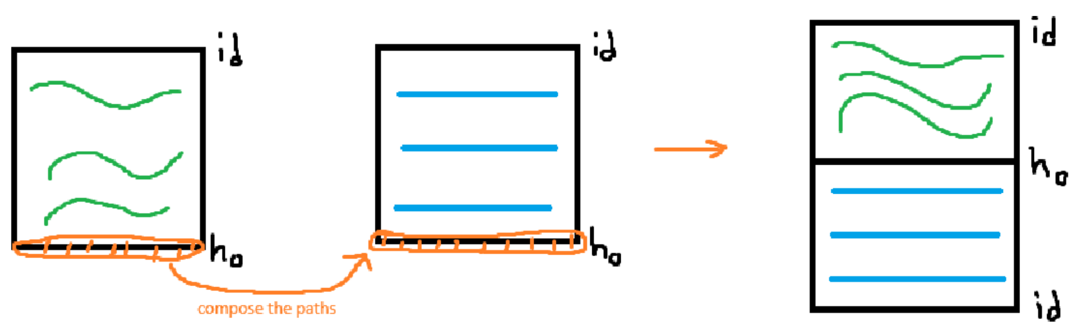
The converse is not true. There is an obstruction. Consider the fibration

$$\mathrm{Diff}_\partial(M \times I) \to \mathscr{C}_{\mathrm{id}}(M) \to \mathrm{Diff}_\partial(M)$$

where the first map is the inclusion and the second is the restriction to the top face.

Lemma. *If any map that is psuedo-isotopic to the identity are isotopic to the identity and the restriction map above has a non-trivial kernel on π_0 then $\pi_0\mathscr{C}_{\mathrm{id}}(M) = 0$.*

Proof. Consider a psuedo-isotopy to the identity, that is an $h_t \in \mathscr{C}_{\mathrm{id}}(M)$. Then there is an isotopy between h_0 and the identity. By composing these two paths in $[M, M \times I]$ we obtain a psuedo-isotopy from the identity to itself, that is an element of $\pi_1\mathrm{Diff}_\partial(M)$.



It is clear that this last square in the illustration has a cube with the back face the identity iff only the top half does (one way to see this is that a path in the concordance space is a cube and if we had the required cube for the green then this would have a blue face on the bottom, thus its kind of like folding the thing on the right in half). Hence the psuedo-isotopy we started with would be contractable iff this loop in $\pi_1 \mathrm{Diff}_\partial(M)$ is. Using the LES in the fibration we can rephrase this condition algebraically as kernels or images vanishing.

Finally Cerf actually proved by independent means that $\pi_0 \mathscr{C}_{\mathrm{id}}(M) = 0$. This is the hard part of the work and the part that needs the extra assumptions that M be simply connected and of dimension ≥ 5.

References

[Cer71] Jean Cerf. The pseudo-isotopy theorem for simply connected differentiable manifolds. In Nicolaas H. Kuiper, editor, *Manifolds — Amsterdam 1970*, pages 76–82, Berlin, Heidelberg, 1971. Springer.